Technical Comments

Comment on "Evaluation of the Computational Errors of Strapdown Navigation Systems"

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A RECENT paper by Sullivan presents a performance evaluation of strapdown systems with ideal inertial instruments.[1] From my own work in this area[2,3] I wish to offer the following additional insights, regarding both pulse-torque/DDA mechanization and the Runge-Kutta scheme used in Ref. 1:

1) Relative merits of DDA or polynomial integration will depend upon environment (the former is sensitive to angular oscillations about a fixed skewed axis,[2] whereas, to the latter, repetitive coning motion is far more detrimental). Also, complete comparative evaluation will include over-all computational requirements for both systems.

2) The concept of coning motion can be generalized beyond the classical force-free precession[4] to include any rotational motion about nonstationary axes.[2] Cumulative coning, in which the rotating axes accumulate several net encirclements about any reference point, occurs when the angular rate about any rotating axis is correlated with the Hilbert transform[5] of the rate about an orthogonal axis (note latter part of the Introduction, Ref. 3). Regardless of whether these cumulative encirclements occur, however, the general angular motion can be characterized by a triad of noisy waveforms allowing any combination of cross-axis correlations (between roll, pitch, and yaw rates and/or their Hilbert transforms).

3) These complex angular motion patterns still lend themselves to accurate analytical prediction of long-term attitude system performance, firmly based on kinematical theory [i.e., using Eq. (7), Ref. 2]. By applying an analogous procedure to the translational motion, reliable predictions can be made for the long-term performance of the navigation system.[3]

4) When the translational motion example of Ref. 1 is placed within the framework of the complete navigation system, another component of accelerometer size effect (specifically, the tangential acceleration component) becomes rectified due to interaction with the attitude system. Appendix A of Ref. 3 shows that this rectified error is of the same order as the centripetal component.

5) With the general (noisy) waveforms anticipated for motion patterns in various practical situations, the roundoff error should take the form of a random walk process. Long-term accumulated error should then be proportional to the square root of the product (average computation rate $\times$ elapsed time).

6) The apparent contradiction in paragraph 1, p. 314 does not represent any basic conflict between the two mechanizations. In characterizing angular motion, a practice found useful by several earlier authors is to separate the direction cosine Taylor expansion into two series [Eq. (3), Ref. 2 was chosen in accordance with this consensus] and to limit system complexity to second order. Once quantization has occurred in the pulsed system, the resulting truncation error is not a performance-limiting factor. In the Runge-Kutta mechanization, the computation algorithm simultaneously affects both the truncation error and the effective resolution. Thus, the improvement in performance with higher-order algorithms and perfect gyros is understandable, and does not violate the principles established by the early authors.

7) For reasons related to the foregoing discussion, the concept of truncation error as described in Ref. 1 has a different significance from that normally associated with DDA mechanizations. The convenience of nomenclature, however, can not eliminate the physical presence of commutation error. On the contrary, it will now be demonstrated that noncommutativity can be blamed for the secular nature of the attitude drift.

For any given time history of vehicle rates, complete over any specified time interval, there is one unique set of outputs which would be seen over this interval by a triad of perfect gyros. When these gyro measurements are sampled, however, the converse is not true; for any triplet of sampled gyro outputs, there must exist an infinity of vehicle rate time histories which could have generated it. Within this infinite set, the range of possible angular orientations at the end of the interval is a measure of the drift error. Even in the trivial case where two of the three angular rates vanish identically, this drift error would not be zero, for two reasons. 1) A finite-order data fit can not be equivalent to the Shannon waveform reconstruction. 2) Even this reconstruction would contain an aliasing error inherent in signals of finite duration (and therefore infinite bandwidth) when sampled at finite rates.[6] For sampling rates above that required by Shannon's theorem, however, the errors from these sources do not increase linearly with time. Despite this, there are certain angular rate patterns (e.g., in-phase correlation between orthogonal rates) producing steadily increasing drifts with DDA integration, and certain patterns (e.g., quadrature correlation between orthogonal rates) which similarly degrade Runge-Kutta integration. These secular drifts would exist even in the absence of any system limitations other than finite sampling rates; yet it has just been explained that linearly time-varying drift cannot be attributed to sampling error in any one channel. The explanation lies in the combined processing of information from more than one axis after part of each waveform fine structure has been sacrificed (whether by sampling or by quantization). The loss of this fine structure corresponds to an angular rate uncertainty; the time integral of this uncertainty closely approximates a finite angle that interacts with the angular rate [see Eq. (7) of Ref. 2] in such a way that may produce a linearly increasing angular orientation error. This of course depends upon 1) whether the uncertainty just described is correlated with the corresponding angular rate or with its Hilbert transform; 2) the correlation of the angular rate in each channel with the angular rate, and with the Hilbert transform of the angular rate, of each other channel; and 3) the relative directions of the angular rate and the uncertainty vectors in Eq. (7) of Ref. 2.

Both Refs. 2 and 3 mention drifts arising from quadrature correlation. They were not carried through the entire system due to the presence of dominating instrument errors and due to speculation as to whether such motion would be a likely occurrence (there is a scarcity of vibration test data with correlation measurements; and space systems, where sustained precession is a natural phenomenon, are often equipped with dampers). If deemed significant, however, this type of error could be traced through the system quite easily. For example, if a slightly misaligned pitch gyro can sense a small fraction of a yaw rate which is correlated with the Hilbert transform of the roll rate, Eq. (7) of Ref. 2 immediately indicates

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a secular yaw drift equal to the product of the misalignment angle and the yaw-roll quadrature covariance, divided by the center frequency of oscillation. (Actually, scale factor errors can in general produce additional secular drift in such an environment, although these errors were dismissed in Ref. 3.) The same general formulation (statistical moments involving the components of a vector cross product) lends itself to evaluation of computational drifts arising from a Runge-Kutta integration algorithm. This calls for establishing the covariance of a Gaussian waveform with the error in an nth-order polynomial fitted to that waveform. Secular drifts produced would be characterized as resulting from combined processing of angular information from different channels while commutativity holds only in the infinitesimal limit.

References

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Comment on "An Empirical Expression for Drag Coefficients of Cones at Supersonic Speeds"

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HILL[1] presents a curve fit for the forebody drag coefficient of sharp cones at zero yaw. The forebody drag coefficient is identically equal to the pressure coefficient for sharp cones when the Mach number is high enough to permit shock

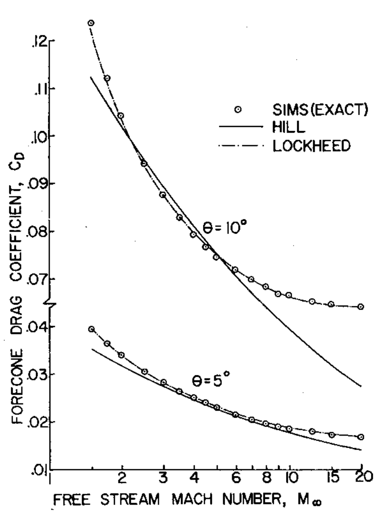

Fig. 1 Variation of forecone drag coefficient with Mach number for sharp cones of half-angle 5° and 10°.

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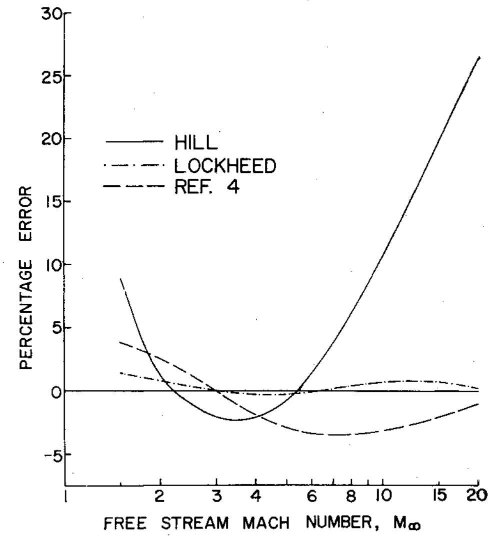

Fig. 2 Percentage error of various theories in predicting forecone drag coefficients for a half-cone angle of 10°.

attachment. The exact or Taylor-Maccoll solution is tabulated in the Sims tables[2] and elsewhere. More accurate curve fits than that of Ref. 1 are available in the literature. The expressions presented by Simon and Walter[3] and Linnell and Bailey[4] appear to be superior to the present work. The Linnell-Bailey formula is particularly simple and is amenable to slide rule as well as machine calculation;

$$C_{P\text{cone}} = (4\sin^2\theta)(2.5 + 8\beta\sin\theta)/(1 + 16\beta\sin\theta)$$

where $\beta = (M^2 - 1)^{1/2}$ and θ is the half-cone angle. Typical errors using this formula are on the order of 2–3% from shock attachment through high hypersonic Mach numbers.

For purposes of machine computation the following formula developed at Lockheed Missiles & Space Company using least-squares techniques is suggested:

$$C_{P\text{cone}} = (2\sin^2\theta)e^x$$

where

$$x = 0.18145 - 0.20923y + 0.09092y^2 + 0.006876y^3 - 0.0062225y^4 - 0.000971y^5$$

and where

$$y = \log_{10}(\beta\sin\theta)$$

The average error is less than 0.75% and the only significant variance occurs near shock detachment. Permissible half-cone angles lie between 2.5° and 30°. Comparative results are presented in Figs. 1 and 2.

Hill's paper presents results at Mach 1. No basically hypersonic formulation can be expected to work in this region. Replacing Mach number by the parameter $(M^2 - 1)^{1/2}$ (unified transonic-hypersonic similarity) will improve his results near shock detachment. Once detachment occurs, however, the inviscid flow on the cone surface is entirely subsonic with the sonic line emanating from the shoulder. The pressure on the cone surface is no longer constant and the similarity breaks down. Characteristically the forecone drag will reach a maximum near shock detachment and will decrease with decreasing Mach number. Several representative curves are shown in the book by Hoerner.[5]

References

[1] Hill, J. C., "An Empirical Expression for Drag Coefficients of Cones at Supersonic Speeds," *AIAA Journal*, Vol. 6, No. 1, Jan. 1968, pp. 165–167.

[2] Sims, J. L., "Tables for Supersonic Flow Around Right